DISSIPATIVE DISTRIBUTED SYSTEMS

Curtainday, October 19, 2001

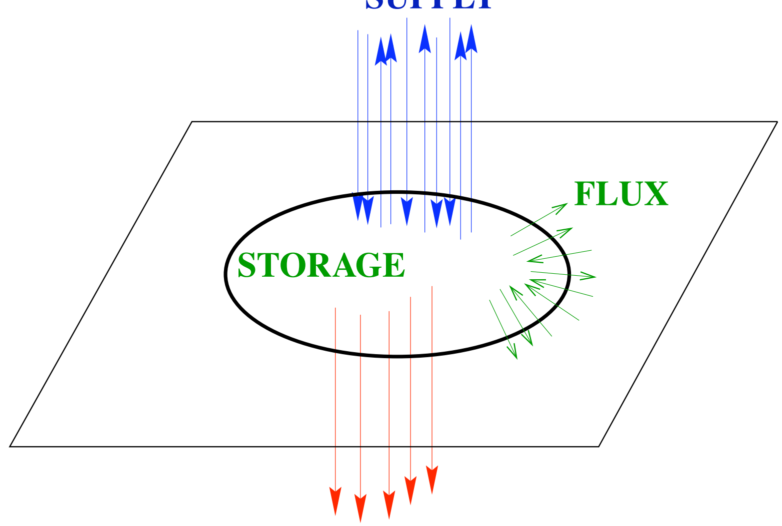
RuG

A **dissipative system** absorbs 'supply', **'globally'** over time and space.

¿¿ Can this be expressed **'locally'**, as

rate of change in storage + spatial flux $\leq$ supply rate

= supply rate + dissipation rate ??



⇒, for 1-D state space systems, the Kalman-Yacubovich-Popov lemma.

⇒ Lyapunov theory for **open** dynamical systems, **LMI's,** their many applications.

⇒ applications in the analysis of open physical systems, synthesis procedures, robust control, passivation control.

Joint work with Harish Pillai (IIT, Mumbay)

& Harry Trentelman.

BEHAVIORAL SYSTEMS

A system **:=** $\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T}$ = **the set of** ***independent*** **variables**

time, space, time and space

$\mathbb{W}$ = **the set of** ***dependent*** **variables**

(= space where the variables take on their values, signal space, space of field variables, . . .)

$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}}$ **:** ***the behavior*** **= the admissible trajectories**

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

for a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,
$w \notin \mathfrak{B}$: the model forbids the trajectory w.

In this lecture, $\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, ('$\mathtt{n}$-D systems') , $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,

$$w : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{w}}, (w_1(x_1, \cdots, x_{\mathtt{n}}), \cdots, w_{\mathtt{w}}(x_1, \cdots, x_{\mathtt{n}})),$$

often, $\mathtt{n} = 4$, independent variables (t, x, y, z),

$\mathfrak{B}$ = solutions of a system of constant coefficient linear PDE's.

'Linear (shift-invariant distributed) differential systems'.

Example: *Maxwell's equations*

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space),

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

n-D LINEAR DIFFERENTIAL SYSTEMS

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,

$\mathfrak{B}$ = the solutions of a linear constant coefficient system of PDE's.

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, and consider

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}$$

$\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ mainly for convenience, but important for some results.

<u>Examples</u>: Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .

Notation for $\mathtt{n}$-D linear differential systems:

$$(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}},$$

Note $\quad \mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})),$

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$$

is called a 'kernel representation' of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$.

THE STRUCTURE OF $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$

PROPERTIES of $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$

- $\exists$ 1 $\leftrightarrow$ 1 relation between $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ and the submodules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$
- **Elimination theorem: the behavior of a subset of the system variables is also described by a PDE**
- **Controllability; image representations**
- **Observability**

Work of Shankar/Pillai, Oberst.

DESCRIBE $(\vec{E}, \vec{j})$ IN MAXWELL'S EQ'NS ?

Eliminate $\vec{B}$, ρ from Maxwell's equations. Straightforward computation yields

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

Elimination theorem $\Rightarrow$ this exercise would be exact & successful.

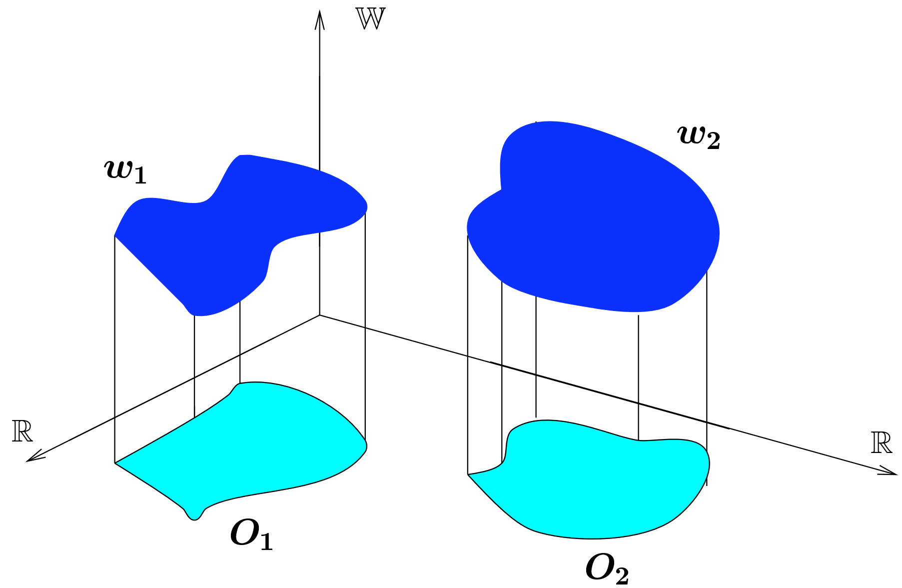
CONTROLLABILITY

Definition: $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **is said to be**

controllable

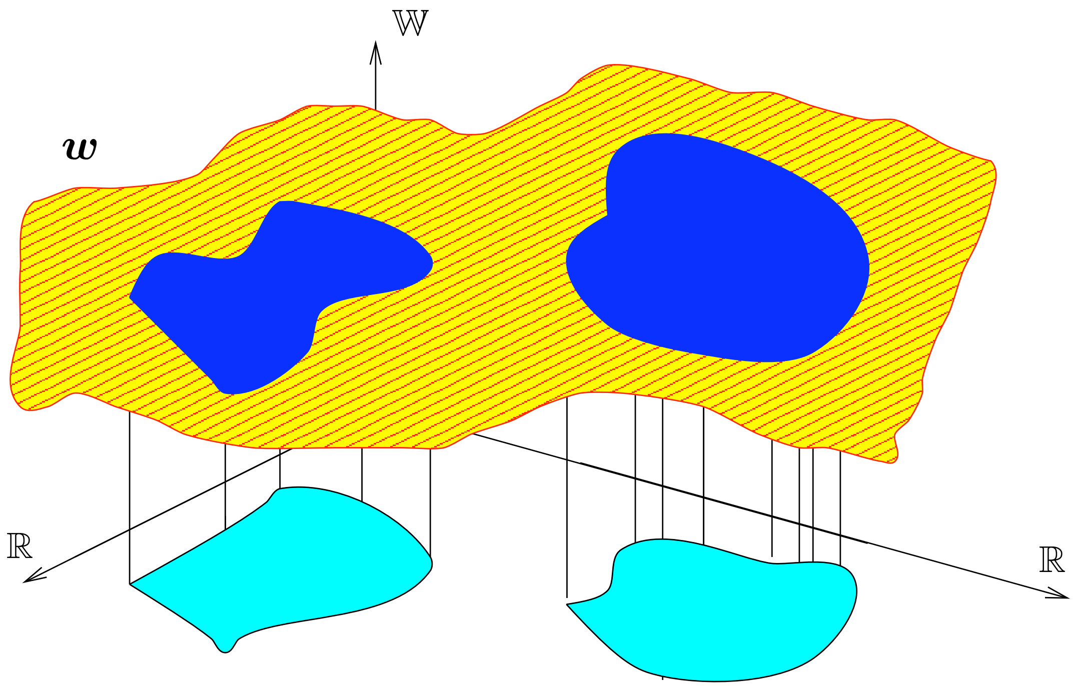
if . . .

In pictures:



$$w_1, w_2 \in \mathfrak{B}.$$

$w \in \mathfrak{B}$ **'patches'** $w_1, w_2 \in \mathfrak{B}$.



Controllability :⇔ 'patch-ability'.

Special case: Kalman controllability for state space systems.

CONDITIONS FOR CONTROLLABILITY

$$R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n})w = 0,$$

is called a *kernel representation* of $\mathfrak{B} = \ker(R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n}))$;

$$w = M(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n})\ell,$$

is called an *image representation* of $\mathfrak{B} = \mathrm{im}(M(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n}))$.

Elimination theorem $\Rightarrow$ **every image is also a kernel.**

¿¿ Which kernels are also images ??

Theorem (Pillai/Shankar):

The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$:

1. **$\mathfrak{B}$ is controllable,**
2. **$\mathfrak{B}$ admits an image representation,**

⋮

ARE MAXWELL'S EQUATIONS CONTROLLABLE ?

The following well-known equations in the ***scalar potential*** $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ **and the** ***vector potential*** $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$**, generate exactly the solutions to Maxwell's equations:**

$$\begin{aligned} \vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla \phi, \\ \vec{B} &= \nabla \times \vec{A}, \\ \vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \phi, \\ \rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi. \end{aligned}$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ a potential!

OBSERVABILITY

$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$ is said to be **observable** if ℓ can be deduced from w, i.e., if $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ is **injective.**

¿¿ Controllability $\Rightarrow \exists$ an observable image representation ??

For 1-D systems, yes!

For n-D systems, not necessarily!

Non-example: **Maxwell's equations. Potential is not observable!**

No potential ever is, for Maxwell's equations.

For n-D systems: image representation requires 'hidden' variables.

DISSIPATIVITY

Multi-index notation:

$x = (x_1, \ldots, x_n)$,

$k = (k_1, \ldots, k_n), \ell = (\ell_1, \ldots, \ell_n)$,

$\xi = (\xi_1, \cdots, \xi_n), \zeta = (\zeta_1, \ldots, \zeta_n), \eta = (\eta_1, \ldots, \eta_n)$,

$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_n}}{\partial x_n^{k_n}})$,

$dx = dx_1 dx_2 \ldots dx_n$,

$R(\frac{d}{dx})w = 0$ for $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$,

etc.

QDF's

The quadratic map in the variables w and their partial derivatives, defined by

$$w \mapsto \sum_{k,\ell} (\frac{d^k}{dx^k} w)^\top \Phi_{k,\ell} (\frac{d^\ell}{dx^\ell} w)$$

is called *quadratic differential form (QDF)* on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.

Here $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$

(given matrices, only finite number $\neq 0$, $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$).

Introduce the 2n-variable polynomial matrix Φ defined by

$$\Phi(\zeta,\eta) = \sum_{k,\ell} \Phi_{k,\ell}\zeta^k\eta^\ell.$$

Denote this QDF as Q_Φ; whence

$$Q_\Phi : w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mapsto Q_\Phi(w) \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}).$$

A QDF is parameterized by a $\Phi(\zeta,\eta) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.

DISSIPATIVE DISTRIBUTED SYSTEMS

We consider only controllable linear differential systems and QDF's.

Definition: $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, **controllable, is said to be** ***dissipative*** **with respect to the supply rate** Q_Φ **(a QDF) if**

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_\Phi(w)\,dx \geq 0$$

for all $w \in \mathfrak{B}$ **of compact support, i.e., for all** $w \in \mathfrak{B} \cap \mathfrak{D}$.

Idea: $Q_\Phi(w)(x_1, \ldots, x_n) dx_1 \cdots dx_n$:
rate of 'energy' delivered to the system.

Dissipativity :$\Leftrightarrow$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w) dx dy dz \right) dt \geq 0 \quad \text{for all } w \in \mathfrak{B}.$$

A dissipative system absorbs net energy (integrated over space and time).

Examples:

Maxwell's eq'ns:

dissipative (in fact, conservative) w.r.t. the QDF $-\vec{E} \cdot \vec{j}$

Passive electrical circuits

Mechanical systems

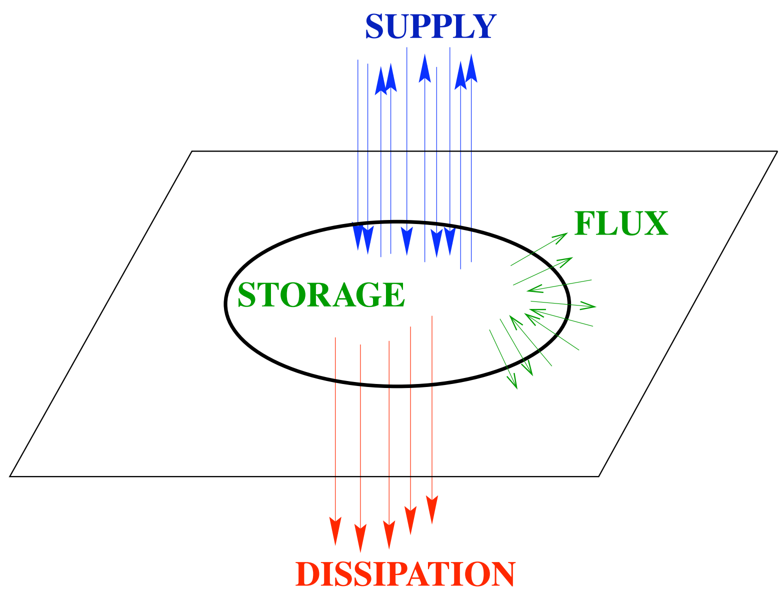
Thermodynamic systems

LOCAL VERSION

Assume that a system is 'globally' dissipative.

¿¿ Can this dissipativity be expressed through a 'local' law??

$$\frac{d}{dt}\ \textbf{Storage} + \textbf{Spatial flux} \leq \textbf{Supply.}$$



Supply = Stored + radiated + dissipated

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0, \quad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D} \qquad \text{'global dissipativity'}$$

$$? \Updownarrow ?$$

$$\frac{\partial}{\partial t} Q_{\Phi_t}(w) + \frac{\partial}{\partial x} Q_{\Phi_x}(w) + \frac{\partial}{\partial y} Q_{\Phi_y}(w) + \frac{\partial}{\partial z} Q_{\Phi_z}(w) \leq Q_\Phi(w)$$

for all $w \in \mathfrak{B}$ **'local dissipativity'.**

$\Leftarrow$: **easy**

$\Rightarrow$: **¡¡ construct $\Phi_t, \Phi_x, \Phi_y, \Phi_z$ from $\mathfrak{B}$ and Φ !!**

THE LOCAL DISSIPATION LAW

Main Theorem: Let $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ be controllable.

Then $\mathfrak{B}$ is **dissipative** with respect to the *supply rate* Q_Φ

iff

there exist

an **image representation** $w = M(\frac{d}{dx})\ell$ of $\mathfrak{B}$, and

an $\mathtt{n}-$**vector of QDF's** $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_{\mathtt{n}}})$

on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{dim}(\ell)})$, called the *flux*,

such that the *local dissipation law*

$$\boxed{\nabla \cdot Q_\Psi(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) that satisfy $w = M(\frac{d}{dx})\ell$.

As usual $\nabla \cdot Q_{\Psi} := \frac{\partial}{\partial x_1} Q_{\Psi_1} + \cdots + \frac{\partial}{\partial x_n} Q_{\Psi_n}$.

Note: the local law involves

(possibly unobservable, - i.e., hidden!) latent variables (the ℓ's).

When the variables are (t, x, y, z), can be reformulated as:

$\exists$ **an image representation** $w = M(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\ell$ **of** $\mathfrak{B}$**,**

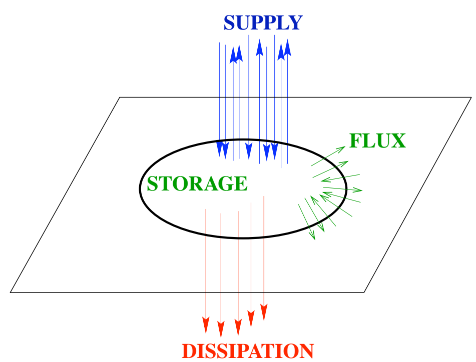
a QDF S**, the** ***storage,*** **and**

a vector of QDF's, $F = (F_x, F_y, F_z)$**, the** ***spatial flux,***

such that

$$\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)$$

holds for all (w, ℓ) **that satisfy** $w = M(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\ell$**.**



EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j}$, the rate of energy supplied.
Introduce the *stored energy density*, S, and
the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S(\vec{E},\vec{B}) = \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) = \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

The following is a local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E},\vec{B}) + \nabla\cdot\vec{F}(\vec{E},\vec{B}) = -\vec{E}\cdot\vec{j}.}$$

Local version involves $\vec{B}$, unobservable from $\vec{E}$ and $\vec{j}$,
the variables in the rate of energy supplied.

IDEA of the PROOF

Consider $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, controllable, and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.
$\mathfrak{B}$ allows an image representation $w = \mathcal{M}(dx)\ell$.
Define $\Phi'(\zeta,\eta) = M^\top(\zeta)\Phi(\zeta,\eta)(M(\eta)$. Note that

$$\boxed{Q_\Phi(w) = Q_{\Phi'}(\ell).}$$

So we may as well study

$$\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)\, dt \quad \text{for } \ell \in \mathfrak{D}$$

instead of

$$\int_{-\infty}^{\infty} Q_\Phi(w)\, dt \quad \text{for } w \in \mathfrak{B} \cap \mathfrak{D}$$

$\rightsquigarrow$ **work with Φ', and 'free' signals ℓ.**

WLOG: $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \ldots$

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow \quad \text{(Parseval)}$$

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n$$

$$\Updownarrow \quad \boxed{\text{(Factorization equation)}}$$

$$\exists\ D : \Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

$$\Updownarrow \quad \text{(easy)}$$

$$\exists\ \Psi, D : \quad \Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - D^\top(\zeta) D(\eta)}{\zeta + \eta}$$

$$\Updownarrow \quad \text{(clearly)}$$

$$\exists\ \Psi : \quad Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{C}^\infty$$

THE FACTORIZATION EQUATION

Consider

$$\boxed{X^\top(-\xi)X(\xi) = Y(\xi)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. *Solvable??*

$\cong$

$$\boxed{X^\top(\xi)X(\xi) = Y(\xi)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown.

Under what conditions on Y does there exist a solution X?

Scalar case: !! write Y as a sum of squares

$$Y = x_1^2 + x_2^2 + \cdots + x_k^2.$$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

For $\mathtt{n} = 1$ and $Y \in \mathbb{R}[\xi]$, solvable (for $X \in \mathbb{R}^2[\xi]$!) iff

$$Y(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

For $\mathtt{n} = 1$, and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that this factorization equation is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

For $\mathtt{n} > 1$, and under these obvious positivity requirements, this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, but it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

This factorizability is a consequence of Hilbert's 17-th problem!

Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, p **given**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ **can in general** **<u>not</u>** **be expressed as as a sum of squares with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$ **(it can for** $n = 1$**).**

But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_n)$, **with** $p(\alpha_1, \ldots, (\alpha_n) \geq 0$, **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, **can** **be expressed as a sum of squares, with the** p_i**'s** $\in \mathbb{R}(\xi_1, \cdots, \xi_n)$, **in fact, with** $k = 2^n$.

This solvability readily leads to solvability of the factorization equation over $\mathbb{R}(\xi_1, \cdots, \xi_{\mathtt{n}})$, for any $\mathtt{n}$.

The need to introduce rational functions (together with the image representation) are the cause of the unavoidable presence of the (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

CONCLUSIONS

- ∃ a nice, algebraic, theory of linear shift-invariant differential systems (PDE's)
- Controllability ⇔ ∃ image representation
- **global dissipation ⇔ ∃ local dissipation law**
- Involves hidden latent variables (e.g. $\vec{B}$ applied to Maxwell's eq'ns)
- The proof ≅ Hilbert's 17-th problem

More info, ms, copy sheets? Surf to

http://www.math.rug.nl/∼willems

HAPPY BIRTHDAY, RUTH !